TWO-SIDED BOUNDS FOR THE VOLUME OF RIGHT-ANGLED HYPERBOLIC POLYHEDRA

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Abstract. For a compact right-angled polyhedron R in $\mathbb{H}^3$ denote by $\mathrm{vol}(R)$ the volume and by $\mathrm{vert}(R)$ the number of vertices. Upper and lower bounds for $\mathrm{vol}(R)$ in terms of $\mathrm{vert}(R)$ were obtained in [3]. Constructing a 2-parameter family of polyhedra, we show that the asymptotic upper bound $5v_3/8$, where v_3 is the volume of the ideal regular tetrahedron in $\mathbb{H}^3$, is a double limit point for ratios $\mathrm{vol}(R)/\mathrm{vert}(R)$. Moreover, we improve the lower bound in the case $\mathrm{vert}(R) \leqslant 56$.

1. Right-angled polyhedra in $\mathbb{H}^3$.

In any space, right-angled polyhedra are very convenient to serve as "building blocks" for various geometric constructions. In particular, they have several interesting properties in hyperbolic 3-space $\mathbb{H}^3$. One can try to obtain a hyperbolic 3-manifold using a right-angled polyhedron as its fundamental polyhedron. Or, one can construct a hyperbolic 3-manifold in such a way that its fundamental group is a torsion-free subgroup of the Coxeter group, generated by reflections across the faces of a right-angled polyhedron [10]. Below we consider only compact polyhedra, which do not admit ideal vertices.

We start by recalling two nice recent results. Inoue [4] introduced two operations on right-angled polyhedra called *decomposition* and *edge surgery*, and proved that Löbell polyhedra (which will be a subject of discussion below) are universal in the following sense:

Theorem 1.1. [4, Theorem 9.1] *Let P_0 be a right-angled hyperbolic polyhedron. Then there exists a sequence of disjoint unions of right-angled hyperbolic polyhedra $P_1, \ldots, P_k$ such that for $i = 1, \ldots, k$, P_i is obtained from P_{i-1} by either a decomposition or an edge surgery, and P_k is a set of Löbell polyhedra. Furthermore,*

$$\mathrm{vol}(P_0) \geqslant \mathrm{vol}(P_1) \geqslant \mathrm{vol}(P_2) \geqslant \ldots \geqslant \mathrm{vol}(P_k).$$

Atkinson [3] estimated the volume of a right-angled polyhedron in terms of the number of its vertices as follows:

Theorem 1.2. [3, Theorem 2.3] *If P is a compact right-angled hyperbolic polyhedron with V vertices, then*

$$(V-2)\cdot\frac{v_8}{32} \leqslant \mathrm{vol}(P) < (V-10)\cdot\frac{5v_3}{8},$$

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where v_8 is the volume of a regular ideal octahedron, and v_3 is the volume of a regular ideal tetrahedron. There is a sequence of compact polyhedra P_i, with V_i vertices such that $\mathrm{vol}(P_i)/V_i$ approaches $5v_3/8$ as i goes to infinity.

A family of polyhedra P_i suggested by Atkinson is described in the proof of [3, Prop. 6.4].

In this note we will demonstrate that Löbell polyhedra can serve as a suitable family realizing the upper bound. Thus these polyhedra play an important role not only in Theorem 1.1, but also in Theorem 1.2.

Let us denote by $\mathrm{vert}(R)$ the number of vertices of a right-angled polyhedron R. In this note we prove that $5v_3/8$ is a double limit point in the sense that it is the limit point of limit points for ratios $\mathrm{vol}(R)/\,\mathrm{vert}(R)$.

Theorem 1.3. *For any integer $k \geqslant 1$ there exists a series of compact right-angled polyhedra $R_k(n)$ in $\mathbb{H}^3$ such that*

$$\lim_{n\to\infty} \frac{\mathrm{vol}(R_k(n))}{\mathrm{vert}(R_k(n))} = \frac{k}{k+1} \cdot \frac{5v_3}{8}.$$

As one will see from the proof, $R_1(n)$ are Löbell polyhedra and $R_k(n)$ for $k > 1$ are towers of them.

Moreover, in Corollary 4.3 we improve the lower estimate from Theorem 1.2 in the case $\mathrm{vert}(R) \leqslant 56$.

2. Löbell polyhedra and manifolds.

We introduced Löbell polyhedra in [10] as a generalization of a right-angled 14-hedron used in [5].

Recall that in order to give a positive answer to the question of the existence of "Clifford-Klein space forms" (that is, closed manifolds) of constant negative curvature, Löbell [5] constructed in 1931 the first example of a closed orientable hyperbolic 3-manifold. This manifold was obtained by gluing together eight copies of the right-angled 14-faced polytope (denoted below by $R(6)$ and shown in Fig. 1) with an upper and a lower basis both being regular hexagons, and a lateral surface given by 12 pentagons, arranged similarly as in the dodecahedron. Obviously, $R(6)$ can be considered as a generalization of a right-angled dodecahedron in the way of replacing basis pentagons to hexagons.

As shown in [10], the dodecahedron and $R(6)$ are part of a larger family of polyhedra. For each $n \geqslant 5$ we consider the right-angled polyhedron $R(n)$ in $\mathbb{H}^3$ with $(2n+2)$ faces, two of which (viewed as the upper and lower bases) are regular n-gons, while the lateral surface is given by $2n$ pentagons, arranged as one can easily imagine. Note that $R(5)$ is the right-angled dodecahedron (see Fig. 1). Existence of polyhedra $R(n)$ in $\mathbb{H}^3$ can be easy checked by involving Andreev's theorem [1].

An algebraic approach suggested in [10] admits a construction of both orientable and non-orientable closed hyperbolic 3-manifolds from eight copies of any bounded right-angled hyperbolic polyhedron. More exactly, any coloring of the faces of a right-angled polyhedron by four colors so that no two faces of the same color share an edge encodes a torsion-free subgroup of orientation preserving isometries which is a subgroup of the polyhedral Coxeter group of index eight. Thus, any four-coloring encodes an orientable hyperbolic 3-manifold obtained from eight

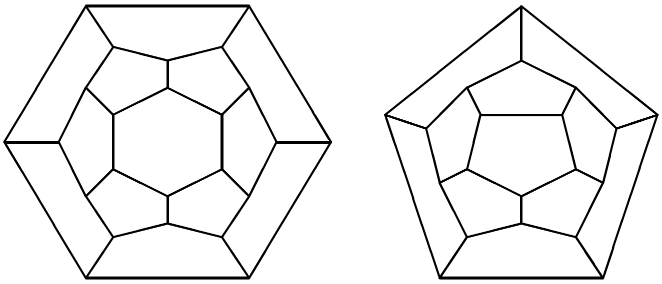

FIGURE 1. Polyhedra $R(6)$ and $R(5)$.

copies of a right-angled polyhedron. This approach also allows one to construct non-orientable hyperbolic 3-manifolds, but in this case five to seven colors are needed.

It was mentioned in [10] that the manifold constructed by Löbell can be encoded by some four-coloring of $R(6)$, and it was shown how to construct concrete orientable and non-orientable manifolds using eight copies of $R(n)$ for any $n \geqslant 5$. Closed orientable hyperbolic 3-manifolds encoded by four-colorings of $R(n)$, $n \geqslant 5$, were called *Löbell manifolds*. (Observe that for each n number of such manifolds do not need to be unique.) Polyhedra $R(n)$ can be naturally referred as *Löbell polyhedra*.

Various properties of Löbell manifolds were intensively studied: the volume formulae were obtained in [9] and [11], invariant trace fields for fundamental groups and their arithmeticity were numerically calculated in [2], many of Löbell manifolds were obtained in [8] as two-fold branched coverings of the 3-sphere, and two-sided bounds for complexity of Löbell manifolds were done in [7].

Since Lobachevsky's 1832 paper, the following *Lobachevsky function* has traditionally been used in volume formulae for hyperbolic polyhedra

$$\Lambda(x) = -\int_0^x \log|2\sin(t)|\,\mathrm{d}t.$$

The volume formula for Löbell manifolds established in [11] implies the following formula for $\operatorname{vol} R(n)$, since any Löbell manifolds indexed by n is glued by isometries from eight copies of $R(n)$:

Theorem 2.1. *For all $n \geqslant 5$ we have*

$$\operatorname{vol}(R(n)) = \frac{n}{2}\left(2\Lambda(\theta_n) + \Lambda\left(\theta_n + \frac{\pi}{n}\right) + \Lambda\left(\theta_n - \frac{\pi}{n}\right) + \Lambda\left(\frac{\pi}{2} - 2\theta_n\right)\right),$$

where

$$\theta_n = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right).$$

It is easy to check that $\theta_n \to \pi/6$ and $\frac{\operatorname{vol} R(n)}{n} \to \frac{5v_3}{4}$ as $n \to \infty$. Here we use that $v_3 = 3\Lambda(\pi/3) = 2\Lambda(\pi/6)$. Moreover, the asymptotic behavior of volumes of Löbell manifolds was established in [7, Prop. 2.10]. This implies trivially the description of the asymptotic behavior of $\operatorname{vol}(R(n))$ as n tends to infinity.

Proposition 2.1. *The following inequalities hold for sufficiently large n:*

$$\frac{5v_3}{4} \cdot n - \frac{17v_3}{2n} < \mathrm{vol}(R(n)) < \frac{5v_3}{4} \cdot n.$$

Since $\mathrm{vert}(R(n)) = 4n$, we get

Corollary 2.1. *The following inequalities hold for sufficiently large n:*

$$\frac{5v_3}{16} - \frac{17v_3}{8n^2} < \frac{\mathrm{vol}(R(n))}{\mathrm{vert}(R(n))} < \frac{5v_3}{16}.$$

3. Proof of Theorem 1.3.

We will use Löbell polyhedra $R(n)$ as building blocks to construct right-angled polyhedra with necessary properties. Let us present polyhedra $R(n)$ by their lateral surfaces as it is done in Fig. 2 for polyhedra $R(6)$ and $R(5)$, keeping in mind that left and right sides are glued together.

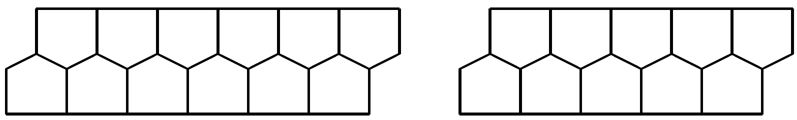

Figure 2. Polyhedra $R(6)$ and $R(5)$.

For integer $k \geq 1$ denote by $R_k(n)$ the polyhedron constructed from k copies of $R(n)$ gluing them along n-gonal faces similar to a tower. In particular, $R_1(n) = R(n)$. The polyhedron $R_3(6)$ is presented in Fig. 3.

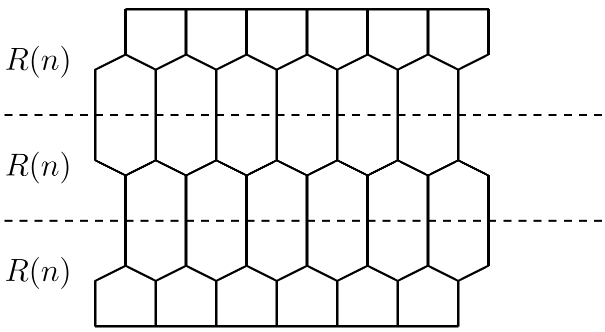

Figure 3. Polyhedron $R_3(6)$.

Obviously, $R_k(n)$ is a right-angled polyhedron with n-gonal top and bottom and the lateral surface formed by $2n$ pentagons and $(k-1)n$ hexagons.

Since $\mathrm{vol}(R_k(n)) = k \cdot \mathrm{vol}(R(n))$, Proposition 2.1 implies that for sufficiently large n

$$k \cdot \frac{5v_3}{4} \cdot n - k \cdot \frac{17v_3}{2n} < \mathrm{vol}(R_k(n)) < k \cdot \frac{5v_3}{4} \cdot n.$$

Since $\mathrm{vert}\, R_k(n) = (2k+2)n$, we obtain

$$\frac{k}{k+1} \cdot \frac{5v_3}{8} - \frac{k}{k+1} \cdot \frac{17v_3}{4n^2} < \frac{\mathrm{vol}(R_k(n))}{\mathrm{vert}(R_k(n))} < \frac{k}{k+1} \cdot \frac{5v_3}{8}.$$

Thus family of right-angled polyhedra $R_k(n)$ is such that for any integer $k \geqslant 1$

$$\lim_{n\to\infty} \frac{\mathrm{vol}(R_k(n))}{\mathrm{vert}(R_k(n))} = \frac{k}{k+1} \cdot \frac{5v_3}{8},$$

and the upper bound $5v_3/8$ is a double limit point in the sense that it is the limit of above limit points as $k \to \infty$:

$$\lim_{k,n\to\infty} \frac{\mathrm{vol}(R_k(n))}{\mathrm{vert}(R_k(n))} = \frac{5v_3}{8}.$$

Thus, the theorem is proved. □

4. Other volume estimates.

Since 1-skeleton of a right-angled compact hyperbolic polyhedron P is a trivalent plane graph, one can easy see that Euler formula for a polyhedron implies

$$V = 2F - 4,$$

where V is number of vertices of P and F is number of its faces. Moreover, Euler formula implies also that P has at least 12 faces (this smallest number of faces corresponds to a dodecahedron). Thus, Theorem 1.2 implies the following result.

Corollary 4.1. *If P is a compact right-angled hyperbolic polyhedron with F faces, then*

$$(F-3) \cdot \frac{v_8}{16} \leqslant \mathrm{vol}(P) < (F-7) \cdot \frac{5v_3}{4}.$$

We recall that constants v_3 and v_8 are

$$v_3 = 3\,\Lambda(\pi/3) = 1.0149416064096535\ldots$$

and

$$v_8 = 8\,\Lambda(\pi/4) = 3.663862376708876\ldots.$$

Since a right-angled hyperbolic n-gon has area $\pi/2 \cdot (n-4)$, the lateral surface area of a compact hyperbolic right-angled polyhedron P with F faces is equal to $\pi \cdot (F-6)$. Thus, Corollary 4.1 implies the following result.

Corollary 4.2. *If P is a compact right-angled hyperbolic polyhedron with lateral surface area S, then*

$$(S/\pi + 3) \cdot \frac{v_8}{16} \leqslant \mathrm{vol}(P) < (S/\pi - 1) \cdot \frac{5v_3}{4}.$$

Observe, that Theorem 2.1 can be used to show that the volume function $\mathrm{vol}\, R(n)$ is a monotonic increasing function of n (see [4] and [7] for proofs), and to calculate volumes of Löbell polyhedra. In particular,

$$\mathrm{vol}\, R(5) = 4.306\ldots, \qquad \mathrm{vol}\, R(6) = 6.023\ldots, \qquad \mathrm{vol}\, R(7) = 7.563\ldots.$$

Together with Theorem 1.1 it gives that the right-angled hyperbolic polyhedron of smallest volume is $R(5)$ (a dodecahedron) and the second smallest is $R(6)$. Thus, if a compact right-angled hyperbolic polyhedron P is differ of a dodecahedron, then

$$\mathrm{vol}(P) \geqslant 6.023\ldots.$$

Thus, we get the following

Corollary 4.3. *If P is a compact right-angled hyperbolic polyhedron different than a dodecahedron, having V vertices and F faces. Then*

$$\mathrm{vol}(P) \geqslant \max\{(V-2)\cdot\frac{v_8}{32}, 6.023\ldots\}$$

and

$$\mathrm{vol}(P) \geqslant \max\{(F-3)\cdot\frac{v_8}{16}, 6.023\ldots\}.$$

The estimates from Corollary 4.3 improve the lower estimate from Theorem 1.2 for $V \leqslant 54$ and the lower estimate from Corollary 4.1 for $F \leqslant 29$.

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